International Journal of Theoretical Physics, Vol. 40, No. 1, 2001

Small-Scale Structure of Space-Time and Dirac Operator[1]

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Received January 25, 2000; revised June 19, 2000

Connes' noncommutative geometry is presented and the relevance of the Dirac operator in the elucidation of the structure of space-time at the Planck length is discussed.

1. INTRODUCTION

One outstanding problem in theoretical physics is the correct marriage of quantum field theory and general relativity. Riemannian geometry provides the right framework for general relativity and for understanding the large-scale structure of space-time. On the other hand, in quantum field theory, the Standard Model has been quite successful in describing all of the phenomenology associated with the elementary particles, consistent with present-day experiments. The fundamental difference between these two theories is that while in the Standard Model, Minkowskian space-time is treated as a background, in general relativity, the metric of space-time is itself a dynamical field. We have learned, however, from quantum field theory that all dynamical objects exhibit a quantum behavior and we are therefore led to conclude that space-time geometry, too, must exhibit a quantum behavior. In the early days of quantum gravity, it was thought that the natural way to achieve this was to quantize the metric of the gravitational field by applying the perturbative techniques that were so successful for the quantization of the other fields in nature. As we now know, the resulting theories failed because they turned out to be nonrenormalizable and/or nonunitary. Thus the question remains as to the appropriate

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0020-7748/01/0100-0139$19.50/0

way of describing the quantum nature of space-time. From a mathematical point of view, this question translates into the corresponding one of how to build a diffeomorphic invariant formulation of the theory which would thus provide a unifying picture of gravitation and the other fundamental interactions in nature.

Present-day attempts to develop a mathematics that goes beyond present quantum field theory involve a variety of approaches, such as topological quantum field theory, dynamical triangulations, string theory and efforts in this context to develop a nonperturbative formulation that could allow us to reach Planck scale physics, noncommutative geometry, and loop representations. The most popular is string theory (in its seemingly duality related variants), which is based on the idea of using extended objects to construct world sheets, thus removing points, which are the source of the singularities, as essential structural objects of the manifold. Here gravitation would appear as a classical limit of the theory. Some of the other approaches are based on the premise that there is nothing wrong with points in a manifold, but that the problem resides in the assumed validity of perturbation theory for quantum gravity. This view has led to a whole line of research intended to develop various techniques in order to construct a nonperturbative quantum gravity theory. For a collection of work on some of these different directions of research see, e.g., Rovelli and Smolin (1995). Although none of these apparently conceptually different approaches and their variants are near a final theory of grand unification, and probably no single one of these directions will succeed in producing it, there appears to be emerging a common denominator of noncommutativity in some of their ingredients. All this seems to point to the fact that when considering the problem of the coordinates below the Planck length, there is no good reason to presume that the texture of space-time will still have a 4-dimensional continuum.

There is a heuristic argument that points in this direction and displays the interplay of quantum mechanics with special and general relativity: Heisenberg's uncertainty principle gives $\triangle x \, \triangle p \sim \hbar$. Thus, a localization $\triangle x$ implies an energy transfer $\sim \hbar c/\triangle x$ and a corresponding inertial mass $m_a = \hbar/(c\triangle x)$. Because of the principle of correspondence,

$$m_a = m_g = \frac{\hbar}{c\triangle x} \tag{1}$$

and this gravitational mass generates in turn a gravitational field. Let us assume that this field is centrally symmetric and given by the Schwarzschild metric

$$ds^2 = \left(c^2 - \frac{2km}{r}\right)dt^2 - r^2(\sin^2\theta \, d\phi^2 + d\theta^2) - \frac{dr^2}{1 - 2km/rc^2},$$

$$k = 6.67 \times 10^{-8} \frac{\text{cm}^3}{\text{g sec}^2}$$

At the Schwarzschild radius, the gravitational mass (1) yields

$$r = \frac{2km}{c^2} = \frac{2k\hbar}{c^3 \triangle x} \Rightarrow \triangle x \cdot r = \frac{2k\hbar}{c^3} \sim \lambda_P^2 = (1.6 \times 10^{-33})^2 \text{ cm}^2$$

The Planck length is a lower limit to the possible precision of measurement of position, and shorter distances do not appear to have an operational meaning. If so, then it would make sense that we need to extended the phase-space noncommutativity of quantum mechanics to a noncommutativity of space-time in order to quantize gravity. A possible way to implement this idea is through a new paradigm of geometric space that would allow us to incorporate into our formalism completely different small-scale structures from those to which we are usually accustomed. One such paradigm is the noncommutative geometry invented by Alain Connes, which has the following features:

1. It includes ordinary Riemannian space.
2. It treats discrete spaces on the same footing as the continuum, thus allowing for a mixture of the two.
3. It allows the possibility of noncommuting coordinates.
4. It is quite different from the geometry arising in string theory, but is not incompatible with the latter.

In order for this paradigm to deserve the name of geometry, it is necessary to consider a new calculus, the so-called spectral calculus, based on operators in Hilbert space and the use of the tools of spectral analysis. This will be the subject matter Sections 2 to 4; Section 5 will be devoted to a discussion of some of the basic ideas behind spectral gravity, exemplified by an application of the general methodology to the simplest noncommutative case of the Einstein–Yang–Mills system with one gauge field. Section 6 is devoted to a discussion of some of the remaining basic problems together with possible future approaches to the issues here presented.

I believe that the solution of the problem of the unification of gravitation and the rest of physics will most probably involve a symbiosis of many of the different present directions of research. I also believe that noncommutative geometry will play an important role in whatever the ultimate theory will be. Thus, without the deliberate intent of dismissing other theories and even of other approaches within noncommutative geometry, I devote the limited space available here to an overview of Connes' noncommutative geometry, including specifically the material that I thought a nonspecialized physicist would find illustrative and motivating and would also require in order to further read the research papers that have been written on spectral gravity. It is interesting, however, that even with an elementary use of the ideas of noncommutative geometry, one obtains many remarkable results, such as the noncommutative origin of the gauge fields. I hope this will justify the content and form of the presentation, as well as the omission of other approaches associated with this quickly developing field. For a discussion of other noncommutative versions of differential geometry see Dubois-Violette (1999) as well as the work of Woronowicz (1987) on differential calculus based on bicovariant bimodules over quantum groups.

The main reference is the treatise of Connes (1994). There are other works of a more introductory nature, such as the lecture notes by Kastler (1998), Schücker (1997), Landi (1997), and Madore (1998).

2. QUANTIZED CALCULUS

At scales below the Planck length, space-time may no longer have the structure of a 4-dimensional continuum, hence the basic idea of noncommutative geometry, of switching from manifolds to algebras where, in general, there is no remanent analogue of space whatsoever. The starting point of Connes, (1994) noncommutative geometry consists in fixing a pair $(\mathcal{H}, F)$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space and F is an operator acting on $\mathcal{H}$ such that $F = F^*$, $F^2 = 1$. Giving F is the same as giving the decomposition of $\mathcal{H}$ into a direct sum of the two infinite-dimensional orthogonal closed subspaces $\{\xi \in \mathcal{H}; F\xi = \pm\xi\}$. The classical differential and integral calculus is then replaced according to the scheme in Table I.

The transitions implied by the first, second, and fifth entries of the table are entirely similar to those required for going from classical to quantum

Table I.

Classical	Quantum
Complex variable	Operator in $\mathcal{H}$
Real variable	Self-adjoint operator in $\mathcal{H}$
Infinitesimal	Compact operator in $\mathcal{H}$
Infinitesimal of order α	Compact operator in $\mathcal{H}$ whose characteristic values μ_n satisfy $\mu_n = O(n^{-\alpha})$, $n \to \infty$
Differential of real or complex variable	$da = [F, a] = Fa - aF$
Integral of infinitesimal of order 1	Dixmier trace

mechanics. This, in turn, implies the fundamental idea of incorporating quantum mechanics into the construction of the geometry from the start, rather than quantizing the geometry *a posteriori*. Also note that in this framework the condition

$$\forall \epsilon > 0, \quad \exists \text{ a finite-dimensional subspace } E \subset \mathcal{H}: \quad \|T_{E_\perp}\| < \epsilon$$

which characterizes compact operators $T \in \mathcal{K}(\mathcal{H})$, can be regarded in a sense as a concept of smallness and thus these operators play the role of infinitesimals. [We designate the bilateral ideal of compact operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$.]

The size of the infinitesimal T is governed by the rate of decay of the sequence $\{\mu_n(T)\}$ as $n \to \infty$, where μ_n are the eigenvalues of $|T| = \sqrt{T^*T}$. Therefore infinitesimals of order $\alpha \in \mathbb{R}^+$ are the two-sided ideals whose elements satisfy the condition

$$\exists C < \infty: \quad \mu_n(T) \leq Cn^{-\alpha}, \qquad \forall n \geq 1$$

Consider the fifth entry in the scheme of Table I, which is the operator-theoretic notion for the differential

$$da = [F, a] \tag{2}$$

where $a \in \mathcal{A}$ (an involutive algebra of operators in the Hilbert space). Since the left side of this equation is to be interpreted as an infinitesimal, we need first to specify the required properties of the representation of $\mathcal{A}$ in the pair $(\mathcal{H}, F)$ such that $[F, a] \in \mathcal{K}$, $\forall a \in \mathcal{A}$. Such a representation is called a Fredholm module and is given by the following definition.

Definition 2.1. An odd Fredholm module over $\mathcal{A}$ is given by:
1. An involutive representation π of $\mathcal{A}$ in $\mathcal{H}$.
2. An $F = F^*$, $F^2 = 1$, such that $[F, \pi(a)]$ is compact for any $a \in \mathcal{A}$.

Definition 2.2. An even Fredholm module is given by $(\mathcal{H}, F)$ as above plus a $\mathbb{Z}_2$ grading $\epsilon = \epsilon^*$, $\epsilon^2 = 1$, of $\mathcal{H}$ such that

$$\epsilon\pi(a) = \pi(a)\epsilon, \quad \forall a \in \mathcal{A}, \qquad \epsilon F = -F\epsilon$$

From here on, when there is no risk of confusion, we shall use for brevity the symbol a to mean the representation $\pi(a)$ of an element of the C^*-algebra $\mathcal{A}$. Finally, we want an 'integral' which neglects all infinitesimals of order >1. In general, however, an infinitesimal of order 1 is not in the domain of the trace (the trace diverges as $\ln N$) and, in addition, the trace of higher order infinitesimals does not vanish. The Dixmier trace is a scale-invariant procedure designed to extract the coefficient of the divergence, thus overcoming these two problems (Dixmier, 1964, 1966),

$$\mathrm{Tr}_\omega(T) = \lim_\omega = \frac{1}{\ln N}\sum_{n=0}^{N-1}\mu_n(T), \qquad \forall T \geq 0 \quad \text{and} \quad T \in \mathcal{L}^{(1,\infty)}$$

(the ideal of compact operators which are infinitesimal of order 1). Here $\lim_\omega$ is a homothetically invariant limit (Dixmier has proven that there exist infinitely many of them) such that the Dixmier trace acquires the following properties:

$$\mathrm{Tr}_\omega\,(\lambda T_1 + T_2) = \lambda \mathrm{Tr}_\omega\,(T_1) + \mathrm{Tr}_\omega\,(T_2), \qquad \forall \lambda \in \mathbb{C}$$

$$\mathrm{Tr}_\omega\,(ST) = \mathrm{Tr}_\omega\,(TS) \qquad \text{for any bouded } S$$

$$\mathrm{Tr}_\omega\,(T) \geq 0 \qquad \text{whenever} \quad T \geq 0$$

$$\mathrm{Tr}_\omega\,(T) = 0 \qquad \text{whenever the order of } T \text{ is } >1$$

As it turns out, for many problems of interest in physics where T is pseudodifferential and measurable, such as is the case for gauge theories and gravitation, the Dixmier trace does not depend on the limiting procedure ω and this common value is the appropriate integral for T in the new calculus. Moreover, in such cases, the Dixmier trace coincides with the Wodzicki residue (Wodzicki, 1984), and we shall use this fact in our calculations later.

3. METRIC IN NONCOMMUTATIVE GEOMETRY

On Riemannian manifolds, the metric is given by the geodesic distance

$$d_\gamma\,(x, y) = \inf_\gamma \{\text{length of paths } \gamma \text{ from } x \text{ to } y\} \tag{3}$$

We will show how Riemannian geometries can be algebraized in order to arrive at a formulation which can be extended to noncommutative spaces. If $\mathcal{A}$ is the algebra of $C^\infty(M)$ functions over M, the above equation can be dualized in the sense of the Gel'fand–Naimark theorem. Let ds be the line element in Riemannian geometry and

$$d(x, y) = \sup\{|f(x) - f(y)|;\, f \in \mathcal{A},\, \left\|\frac{df}{ds}\right\| \leq 1\} \tag{4}$$

To show that (3) and (4) agree for M a compact Riemannian space, recall that a function is Lipschitz if

$$|f(x) - f(y)| \leq C d_\gamma(x, y) \qquad \forall x, y \in M \tag{5}$$

$$\|f\|_{\mathrm{Lip}} = C = \sup \frac{|f(x) - f(y)|}{d_\gamma\,(x, y)} \tag{6}$$

The algebra of Lipschitz functions is norm-dense in the algebra of continuous functions on M, so

$$\left\|\frac{df}{ds}\right\| = \|f\|_{\mathrm{Lip}}$$

The condition $\|df/ds\| \leq 1$ implies that

$$d_\gamma(x, y) \geq \sup |f(x) - f(y)| = d(x, y) \tag{7}$$

To invert the inequality, fix y and consider the function $f_{\gamma,y}(p) = d_\gamma(p, y)$. Then $\|df_{\gamma,y}/ds\| \leq 1$, and from (4),

$$\begin{aligned} d(x, y) &\geq |f_{\gamma,y}(x) - f_{\gamma,y}(y)| = |d_\gamma(x, y) - d_\gamma(y, y)| \\ &= d_\gamma(x, y) \end{aligned} \tag{8}$$

From (7) and (8), we get

$$d_\gamma(x, y) = d(x, y)$$

so (4) does in fact yield the geodesic distance between any two points. To measure distances in a possibly noncommutative space X, we generalize (4) by specifying a metric structure on X. We thus define a "unit of length" by an operator of the form

$$G = \sum_1^q (dx^\mu)^* \, g_{\mu\nu} \, (dx^\nu)$$

where x^μ are elements of $\mathcal{A}$, $dx^\mu = [F, x^\mu]$, and $g = g_{\mu\nu}$, $\mu, \nu = 1, \ldots, q$, is a positive element of the matrix algebra $M_q(\mathcal{A})$. Note that $G \in \mathcal{K}$ and that it is a positive "infinitesimal" by construction. We can therefore think of its positive square root as the line element of Riemannian geometry, i.e.,

$$G^{1/2} = ds \tag{9}$$

Replace the points $x, y \in X$ by the pure states ϕ, χ on the C^*-algebra closure of $\mathcal{A}$, and use the evaluation map by the Gel'fand–Naimark theorem,

$$\phi(a) = a(x), \qquad \chi(a) = a(y), \qquad \forall a \in \mathcal{A} \tag{10}$$

together with the quantum-theoretic expression for the $dx = [F, x]$. With the additional assumption that G commutes with F so that $dG = 0$ (thus avoiding operator ordering ambiguities), we can rewrite the formula (4) as

$$d(\phi, \chi) = \sup\left\{|\phi(a) - \chi(a)|; a \in \mathcal{A}, \left\|\left[\frac{F}{G^{1/2}}, a\right]\right\| \leq 1\right\} \tag{11}$$

The following operator on $\mathcal{H}$ is self-adjoint:

$$D := \frac{F}{G^{1/2}} = F(ds)^{-1} \tag{12}$$

We can then reformulate (11) as

$$d(\phi, \chi) = \sup\{|\phi(a) - \chi(a)|; a \in \mathcal{A}, \|[D, a]\| \leq 1\}$$

This will be then the expression for measuring distances in the spectrum of $\mathcal{A}$ for any pair ϕ, χ of states on $\mathcal{A}$ (commutative or not). By squaring the defining equation (12) for the operator D and making use of the properties of F, we get

$$D^2 = FG^{-1/2}FG^{-1/2} = F^2G^{-1} = G^{-1}, \qquad D = F|D|, \qquad |D| = G^{-1/2} \tag{13}$$

Hence F is by construction the sign of D, and since G is also given in terms of D, the information on the metric structure of our Fredholm module is contained in the self-adjoint unbounded operator D on $\mathcal{H}$. Therefore it turns out more economical to take as our basic data the triple $(\mathcal{A}, \mathcal{H}, D)$.

4. SPECTRAL GEOMETRY

Definition 4.1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra of operators $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D = D^*$ on $\mathcal{H}$ such that:

a. The resolvent $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, of D is compact.

b. The commutators $[D, a]$ are bounded $\forall a \in \mathcal{A}$ (a is then said to be Lipschitz).

We reformulate the quantized calculus discussed in Section 2 in terms of the triple $(\mathcal{A}, \mathcal{H}, D)$. The triple is said to be even if there is a $\mathbb{Z}_2$ grading of $\mathcal{H}$, namely an operator ϵ on $\mathcal{H}$ with $\epsilon = \epsilon^*$, $\epsilon^2 = 1$, such that

$$\epsilon D + D\epsilon = 0, \qquad \epsilon a - a\epsilon = 0, \qquad \forall a \in \mathcal{A}$$

If such a grading does not exist, the triple is said to be odd. In general, one could ask that condition b should be satisfied only for a dense subalgebra of $\mathcal{A}$.

Proposition 4.1. Given a compact operator T on $\mathcal{H}$, its spectrum $\sigma(T)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Furthermore, any nonzero $\lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity.

Because of this, the assumptions in Definition 4.1 of compactness of $(D - \lambda)^{-1}$ imply that its eigenvalues $\mu_n((D - \lambda)^{-1}) \to 0$ as $n \to \infty$. Hence $\mu_n(|D|) \to \infty$ as $n \to \infty$.

4.1. Distance and Integral for a Spectral Triple

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ may be in general a noncommutative algebra, we showed at the end of Section 3 that the natural distance function of the space $\mathcal{S}(\overline{\mathcal{A}})$ of states on the C^*-algebra $\overline{\mathcal{A}}$ was given by

$$d(\phi, \chi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \chi(a)|; \|[D, a]\| \leq 1\}, \qquad \forall \phi, \chi \in \mathcal{S}(\overline{\mathcal{A}}) \tag{14}$$

Now, in order to define the analogue of the classical measure integral, and recalling the discussion of infinitesimals in Section 2, we use the following definition.

Definition 4.1.1. We say that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is of dimension $n > 0$ (or n summable) if $|D|^{-1}$ is an infinitesimal of order $1/n$, i.e.,

$$\exists\, C < \infty: \quad \mu_m (|D|^{-1}) \leq C m^{-1/n}, \quad \forall m \geq 1 \tag{15}$$

where μ_m are the eigenvalues of $|D|^{-1} \in \mathcal{K}(\mathcal{H})$. For such an n-dimensional spectral triple, the integral of any $a \in \mathcal{A}$ is defined by

$$\int a = \frac{1}{C} \operatorname{Tr}_\omega(a|D|^{-n}) \tag{16}$$

Note that here $\mu_m(|D|^{-n}) \leq C m^{-1}$, so $|D|^{-n} \in \mathcal{L}^{(1,\infty)}$, which in turn implies that $a|D|^{-n} \in \mathcal{L}^{(1,\infty)}$. Also note from (15) that the normalization constant in (16) is determined by the behavior of the characteristic values of $|D|^{-n}$. If $|D|^{-n}$ is measurable, then the Dixmier trace will be independent of $\lim_\omega$ and could be computed by the local formula for the Wodzicki residue. To make the integral (16) a nonnegative (normalized) trace on $\mathcal{A}$, we need to introduce the concept of tameness of $\mathcal{A}$, which implies imposing the requirements

$$\int ab = \int ba, \quad \forall a, b \in \mathcal{A}, \qquad \int a^* a \geq 0, \quad \forall a \in \mathcal{A}$$

A particular example of a spectral triple is the so-called canonical triple, where the operator D is the Dirac operator and $\mathcal{A}$ is the commutative algebra of smooth functions on a Riemmanian manifold. We shall discuss this case next, both because it serves to exemplify some of the concepts in the above abstract construction, and also because we shall be using this canonical triple in our discussion of spectral gravity in the last part of this presentation.

4.2. The Canonical Triple over a Manifold

Let (M, g) be a closed n-dimensional Riemannian spin manifold. We define the canonical spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as follows:

1. $\mathscr{A} = C^\infty(M)$, the algebra of complex-valued smooth functions on M.
2. $\mathscr{H} = L^2\,(M, S)$, the Hilbert space of square-integrable sections of the irreducible spinor bundle over M, with rank $2^{[n/2]}$. The scalar product in $L^2(M, S)$ is the natural one in $\mathbb{C}^{[n/2]}$, and the elements of the algebra $\mathscr{A}$ act as multiplicative operators on $\mathscr{H}$.
3. D is the Dirac operator of a Clifford connection.

Consider the algebra morphism $\gamma\colon \Gamma(M, C(M)) \to \mathscr{B}(\mathscr{H})$, where $C(M)$ is the Clifford bundle over M whose fiber at $x \in M$ is the complexified Clifford algebra and $\Gamma(M, C(M))$ is the module of sections. We define $D = \gamma \circ \nabla$, where ∇ is the covariant derivative in the spinor bundle, compatible with the Levi-Civita connection, $\nabla = \nabla_\mu\, dx^\mu$, $\gamma(dx^\mu) =: \gamma^\mu(x)$, $D = \gamma^\mu\, \nabla_\mu$. Let $\mathbf{l}_\alpha$ be an arbitrary basis for the Dirac spinors $\psi \in \mathscr{S}_4(M)$, so that $\psi(x) = \psi^\alpha(x)\mathbf{l}_\alpha$. Let $\blacktriangle$ denote the symplectic spinor product. From the properties of the covariant derivative we get (Luehr and Rosenbaum, 1974)

$$\nabla_\mu\psi(x) = (\partial_\mu\psi^\alpha(x) + \psi^\beta(x)\Lambda^\alpha_{\mu\beta})\mathbf{l}_\alpha, \qquad \Lambda^\alpha_{\mu\beta} = \mathbf{l}^\alpha\blacktriangle\nabla_\mu\mathbf{l}_\beta \tag{17}$$

For the canonical triple. $(\mathscr{A}, \mathscr{H}, D)$ over M, it follows from the Gel'fand–Naimark theorem that the space M is the structure space $\hat{\mathscr{A}}$ of the norm closure $\overline{\mathscr{A}}$ of the pointwise convergence on a commutative C^*-algebra with unit $\mathscr{A}$, namely the space of equivalent classes of irreducible representations of $\mathscr{A}$. Moreover, since $\mathscr{A}$ is Abelian, every irreducible representation is one-dimensional, so $\hat{\mathscr{A}}$ is therefore a *-linear functional $\phi\colon \mathscr{A} \to \mathbb{C}$ which is multiplicative: $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in \mathscr{A}$. It also follows that $\phi(1) = 1$, $\forall\phi \in \hat{\mathscr{A}}$. Thus, the space $\hat{\mathscr{A}}$ is the space of characters of $\mathscr{A}$. The space $\hat{\mathscr{A}}$ is made into a topological space (the Gel'fand space of $\mathscr{A}$) by endowing it with the Gel'fand topology of pointwise convergence on $\mathscr{A}$, and since the algebra $\mathscr{A}$ has a unit, $\hat{\mathscr{A}}$ is a compact Hausdorff space. From (14), we have that the natural distance for M is given by

$$\begin{aligned} d(\phi_x, \phi_y) &= \sup_{a\in\mathscr{A}} \{|\phi_x(a) - \phi_y(a)|;\ \|[D, a]\| \le 1\}; \quad \forall\phi_x, \phi_y \in \hat{\mathscr{A}} \\ &= \sup_{a\in\mathscr{A}} \{|a(x) - a(y)|;\ \|[D, a]\| \le 1\} \end{aligned}$$

The commutator $[D, a]$ is a multiplicative operator since $[D, a]\psi = (\gamma^\mu\partial_\mu a)\psi = \gamma(da)\psi$, so $\|[D, a]\| = \|\gamma(da)\|$. This, together with the fact that $[D, a]$ is bounded iff a is almost everywhere equal to a Lipschitz function f [cf (5)], recovers the geodesic distance given by (4).

We show that the integration given by (16) yields the usual Riemann measure on M,

$$\int_M a = c(n)\, \mathrm{Tr}_\omega(a|D|^{-n}), \qquad c(n) = 2^{(n-[n/2]-1)}\, n\pi^{n/2}\Gamma\left(\frac{n}{2}\right), \qquad \forall a \in \mathcal{A}$$

Locally, the Dirac operator can be written as $D = \gamma(dx^\mu)\partial_\mu$ + lower order terms. Thus, D is elliptic with principal symbol (Taylor, 1981) $\sigma^D(\xi) = \gamma(\xi)$, $\gamma(\xi)^2 = -\|\xi\|^2 1_{2^{[n/2]}}$, or $\gamma(\xi)(-\gamma(\xi)/\|\xi\|^2) = 1_{2^{[n/2]}}$, $(\sigma^D(\xi))^{-1} = -\gamma(\xi)/\|\xi\|^2$. Consequently, $|D|^{-n}$ is a pseudodifferential of order $-n$. But by Connes' theorem, which relates the Dixmier trace to the Wodzicki residue (Wodzicki, 1984; Manin, 1979; Guillemin, 1985),

$$\mathrm{Tr}_\omega(T) = \mathrm{Res}_W(T) =: \frac{1}{n(2\pi)^n}\int_{S^\star M} \mathrm{Tr}_E\, \sigma_{-n}(T)\, d\mu \tag{18}$$

where $S^\star M$ = (the unit cosphere) = $\{(x, \xi) \in T^\star M: \|\xi\| = 1\} \subset T^\star M$, with measure $d\mu = dx\, d\xi$, and Tr_E denotes the matrix trace over the 'internal indices'. Equation (18) becomes

$$\begin{aligned}
\mathrm{Tr}_\omega(a|D|^{-n}) &= \mathrm{Tr}_\omega(a(D^\star D)^{-n/2}) = \mathrm{Tr}_\omega\, (a(D^2)^{-n/2}) = \mathrm{Res}_W(a(D^2)^{-n/2}) \\
&= \frac{1}{n(2\pi)^n}\int_{S^\star M} \mathrm{Tr}\, (a\sigma_{-n}(D^2)^{-n/2})\, dx\, d\xi \\
&= \frac{1}{n(2\pi)^n}\int_{S^\star M} \mathrm{Tr}\, (a\|\xi\|^{-n} 1_{2^{[n/2]}}) dx\, d\xi \\
&= \frac{2^{[n/2]}}{n(2\pi)^n}\int_{S^{n-1}} d\xi \int_M a(x)\, dx
\end{aligned}$$

since on the cosphere bundle $\|\xi\| = 1$. But

$$\int_{S^{n-1}} \xi = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

so

$$\mathrm{Tr}_\omega(a|D|^{-n}) = \frac{\pi^{-n/2}\, 2^{([n/2]+1-n)}}{n\Gamma(n/2)}\int_M a(x)\, dx \quad \blacksquare$$

4.3. Real Structure on a Spectral Triple

This notion (Connes, 1995) is essential in order to introduce the concept of Poincaré duality, and may be thought of as a generalized charge conjugation operator which allows one to keep track of the real structure on $\mathcal{H}$ and plays a crucial role in the derivation of the Lagrangians for gauge fields, as we shall see in Section 5.

Definition 4.3.1. Let $(\mathcal{A}, \mathcal{H}, D)$ be an even spectral triple. A real structure for a 4-dimensional space is an antilinear isometry J in $\mathcal{H}$ such that

$$J^\star = J^{-1} = -J, \quad J^2 = -1, \quad JD = DJ, \quad J\epsilon = \epsilon J \tag{19}$$

$$[a, b^o] = 0, \quad b^o = Jb^\star J^\star \quad \text{for any} \quad a, b \in \mathcal{A} \tag{20}$$

$$[[D, a], b^o] = 0 \tag{21}$$

Condition (21) may be seen to mean that D is a "generalized differential operator" of order 1. From condition (20) and the Jacobi identity, one can show that (21) is equivalent to $[[D, b^o], a] = 0$ for any $a, b \in \mathcal{A}$. Condition (20) also turns $\mathcal{H}$ into a bimodule over $\mathcal{A}$. The bimodule structure is given by

$$a\xi b =: aJb^\star J^\star \xi; \qquad a, b \in \mathcal{A} \tag{22}$$

This bimodule structure follows from the fact that the existence of a J satisfying (20) implies that

$$aJb^\star J^\star \xi = Jb^\star J^\star a\xi$$

Applying (22) to both sides of this equation results in

$$aJb^\star J^\star \xi = a\xi b = Jb^\star J^\star a\xi = a\xi b$$

which shows the consistency of the definition (22). Thus, if $a \in \mathcal{A}$ acts on $\mathcal{H}$ as a left multiplication operator, then $Ja^\star J^\star$ is the corresponding right action.

Observe that for commutative algebras these two actions can be identified and one simply writes $a = Ja^\star J^\star$. Under these circumstances, (21) becomes $[[D, a], b] = 0$, which indeed implies that D is a differential operator of order 1. Furthermore, for a Riemannian spin manifold, the antilinear isometry J is given by $J\psi = C\bar{\psi}$, $\forall \psi \in \mathcal{H}$, where C is the charge conjugation operator.

4.4. Product of Two Real Spectral Triples

If we are given two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1, J_1,)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, J_2)$, where the first one is taken to be even with $\mathbb{Z}_2$ grading ϵ_1 on $\mathcal{H}$, the product triple is $(\mathcal{A}, \mathcal{H}, D, J)$ with

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2,$$
$$D = D_1 \otimes 1 + \epsilon_1 \otimes D_2, \qquad J = J_1 \otimes J_2$$

The dimension of the product triple is the sum of the dimensions of the first and second triples. Also, as expected, two triples are equivalent if $\exists U$: $\mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(a)U^* = \pi_2(a)$ for any $a \in \mathcal{A}_1$, $UD_1U^* = D_2$, $U\epsilon_1 U^* = \epsilon_2$, and $UJ_1U^* = J_2$, where U is unitary.

5. SPECTRAL GRAVITY

Our presentation so far contains the bare bones of Connes' program for noncommutative geometry. Nothing was said about the construction of the higher order calculus starting from the infinitesimals $[D, a]$. This would be basic for gauge theories. Notwithstanding these omissions, the material covered is sufficient to consider a novel approach for studying the coupling of gravitation with gauge fields and fermions, known as spectral gravity. In this section we shall concentrate on a toy model for the bosonic sector of a GUT.

To describe the dynamics of gravitation coupled to gauge degrees of freedom, Chamseddine and Connes (1996) propose the 'purely geometric action'

$$S_B(D, A) = \mathrm{Tr}_{\mathcal{H}}\left(\chi\left(\frac{D_A^2}{\Lambda^2}\right)\right) \tag{23}$$

where D_A is related to the Dirac operator in a way to be discussed below, $\mathrm{Tr}_{\mathcal{H}}$ denotes the usual trace in the Hilbert space, Λ is a "cutoff" parameter, and χ is a suitable function which cuts off all eigenvalues of D_A^2 larger than Λ^2. The motivation for proposing such an action, which depends solely on the spectrum of the self-adjoint operator D_A, resides in a new interpretation (Connes, 1996) of the gauge degrees of freedom as the inner fluctuations of a noncommutative geometry. To account for these fluctuations, the operator D, which gives the 'external geometry', is replaced by $D_A = D + A + JAJ^*$, where A is the gauge potential and J is the real structure antilinear operator discussed above. To understand this important observation, recall first that if M is a smooth (paracompact) manifold, the group $\mathrm{Diff}(M)$ is isomorphic to the group (under map composition) $\mathrm{Aut}(C^\infty(M))$ of $*$-preserving automorphisms of the algebra $C^\infty(M)$. Indeed, let $\alpha \in \mathrm{Aut}(C^\infty(M))$ be an invertible map from $C^\infty(M)$ into itself such that $\alpha(fg) = \alpha(f)\alpha(g)$ and $\alpha(f^*) = (\alpha(f))^*$ for any $f, g \in C^\infty(M)$, and let $\varphi \in \mathrm{Diff}(M)$ be such that

$$\mathbb{C} \ni f(x) = f'(x'), \qquad x' = \varphi \circ x, \qquad f'(x') = f \circ \varphi^{-1}(x') \Rightarrow f' = f \circ \varphi^{-1}$$

Then the relation between φ and the corresponding automorphism α_φ is given via the pullback $\alpha_\varphi(f)(x) = f \circ \varphi^{-1}(x)$, $\forall f \in C^\infty(M)$, $x \in M$. If $\mathcal{A}$ is a unital noncommutative algebra, we define the group $\mathrm{Aut}(\mathcal{A})$ exactly in the same way as above, together with the condition $\beta(1) = 1$ for any $\beta \in \mathrm{Aut}(\mathcal{A})$. Let $u \in \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A};\, uu^* = u^*u = 1\}$. Then an inner automorphism $\alpha_u \in \mathrm{Aut}(\mathcal{A})$ is given by

$$\alpha_u(a) = uau^*, \qquad \forall a \in \mathcal{A}$$

It follows that $\alpha_{u^*} \circ \alpha_u = \alpha_u \circ \alpha_{u^*} = 1_{\mathrm{Aut}(\mathcal{A})}$ for any $u \in \mathcal{U}(\mathcal{A})$, and $\alpha_{u_1} \circ \alpha_{u,_2} = \alpha_{u_1u_2}$, so $\mathrm{Inn}(\mathcal{A}) \subseteq \mathrm{Aut}(\mathcal{A})$ is a normal subgroup. Note also that since

β is *-preserving, $\beta(u)(\beta(u))^* = \beta(u)(\beta(u^*) = \beta(uu^*) = \beta(1) = 1$ and, in analogy, one shows also that $(\beta(u))^*\beta(u) = 1$. Thus, $\beta(u) \in \mathcal{U}(\mathcal{A})$, so any automorphism will preserve the group of unitaries in $\mathcal{A}$. Moreover,

$$\begin{aligned}\alpha_{\beta(u)}(a) &= \beta(u)a\beta(u^*) = \beta(u)\beta(\beta^{-1}(a))\beta(u^*)\\ &= \beta(u\beta^{-1}(a)u^*) = (\beta \circ \alpha_u \circ \beta^{-1})(a)\end{aligned}$$

so $\alpha_{\beta(\mathrm{u})} = \beta \circ \alpha_u \circ \beta^{-1} \in \mathrm{Inn}(\mathcal{A})$, $\forall \beta \in \mathrm{Aut}(\mathcal{A})$ β $_u \in \mathrm{Inn}(\mathcal{A})$ Denoting Out $(\mathcal{A}) =: \mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$, we get the following short exact sequence of groups:

$$1_{\mathrm{Aut}(\mathcal{A})} \to \mathrm{Inn}(\mathcal{A}) \xrightarrow{\sigma} \mathrm{Aut}(\mathcal{A}) \xrightarrow{\rho} \mathrm{Out}(\mathcal{A}) \to 1_{\mathrm{Aut}(\mathcal{A})} \tag{24}$$

If $\mathcal{A}$ is commutative [e.g., $\mathcal{A} = C^\infty(M)$], then $\alpha_u(a) = uau^* = auu^* = a$, $\forall_a \in \mathcal{A}$; all inner automorphisms are then trivial and $\mathrm{Aut}(\mathcal{A}) \equiv \mathrm{Out}(\mathcal{A}) \simeq \mathrm{Diff}(M)$. The above argument leads to the previously announced interpretation that $\mathrm{Inn}(\mathcal{A})$ will give 'internal' gauge degrees of freedom and $\mathrm{Out}(\mathcal{A})$ will give 'external' diffeomorphisms'. Furthermore, the gauge degrees of freedom occur as a consequence of the noncommutativity of the geometry. To prove this, note first that given an irreducible representation π of $\mathcal{A}$ on $\mathcal{H}$, an inner automorphism $\alpha_u \in \mathrm{Inn}(\mathcal{A})$ on the real triple $(\mathcal{A}, \mathcal{H}, D, J)$ will produce a new representation $\pi_u =: \pi \circ \alpha_u$. It can be shown that the triples $(\mathcal{A}, \mathcal{H}, D, J)$ and $(\mathcal{A}, U\mathcal{H}, D_u = D + u[D, u^*] + Ju[D, u^*]J, -J)$ are equivalent through α_u, with the intertwinner unitary given by

$$U = uJuJ^* \tag{25}$$

The proof is straightforward and is based on repeated use of (19) and (20),

$$UJU^* = -J \tag{26}$$

$$U\pi(a)U^* = u\pi(a)u^* = \pi \circ \alpha_u \tag{27}$$

$$UDU^* = D + u[D, u^*] + Ju[D, u^*]J^* := D_u \tag{28}$$

Combining (26)–(28), we arrive at the desired result. ∎

We thus have Connes' beautiful interpretation that the operator D_u can be seen as a consequence of the perturbation of the 'geometry' given by D by the inner (gauge) degrees of freedom given by the potential $A = u[D, u^*]$.

Consider a general internal perturbation of the geometry expressed by

$$D_A = D + A + JAJ^* \tag{29}$$

where A is an arbitrary Hermitian *gauge potential* operator, $A^* = A$, of the form

$$A = \sum_j a_j\,[D, b_j], \qquad a_j, b_j \in \mathcal{A} \tag{30}$$

This will then be the generalized Dirac operator to be used in the right-hand side of the spectral action given in (23). However, before considering the details of such an action, we shall first show that it is invariant under inner automorphisms, i.e., that $D_{A_u} = UD_AU^*$, where U is the unitary operator given by (25). This follows immediately by observing first that

$$\begin{aligned} UD_AU^* &= U(D + A + JAJ^*)U^* \qquad (31)\\ &= D + u[D, u^*] + Ju[D, u^*]J^* + UAU^* + UJAJ^*U^* \quad \text{[by (28)]} \end{aligned}$$

Moreover, recalling in addition (30), we have

$$[A, Ju^*J^*] = \sum_j [a_j\,[D, b_j], Ju^*J^*] = 0 \quad \text{[by (21)]}$$

and

$$\begin{aligned} UAU^* &= uJuJ^*AJu^*J^*u^* \\ &= uJuJ^*Ju^*J^*Au^* = uAu^* \end{aligned} \tag{32}$$

and, by repetitive application of (20), we also get

$$UJAJ^*U^* = JuAu^*J^* \tag{33}$$

Inserting (32) and (33) into (31) results in

$$UD_A\,U^* = D_u + uAu^* + JuAu^*J^* = D_u + A_u + JA_uJ^* = D_{A_u}$$

where $A_u =: uAu^*$. Thus, under inner automorphisms,

$$D_A \to D_{A_u} = UD_AU^* \Rightarrow D^2_{A_u} = UD_AU^*UD_AU^* = UD^2_AU^*$$

$$\mathrm{Tr}(UD^2_AU^*) = \mathrm{Tr}(D^2_A)$$

so (23) is indeed (gauge) automorphism-invariant. ■

For a commutative algebra, the general internal perturbation of the geometry given by (29) does in fact vanish. To this end, recall (see Section 4.3) that for a commutative algebra, right and left actions on $\mathcal{H}$ as a bimodule can be identified, that is, $b = Jb^*J^*$. Consequently,

$$\begin{aligned} J[D, b_j]J^* &= JDb_jJ^* - Jb_jDJ^* = JDJ^*\,Jb_j\,J^* - Jb_jJ^*\,JDJ^* \\ &= DJb_jJ^* - Jb_jJ^*D = [\mathrm{D}, Jb_jJ^*] \\ &= [D, b_j^*] \text{ [by (19)]} \end{aligned}$$

$$
\begin{aligned}
JAJ^* &= J\sum_j a_j[D, b_j]J^* = a_j^* \sum_j [D, b_j^*] = [\text{from (21)}] \\
&= \sum_j [D, b_j^*]a_j^* = -\left(\sum_j a_j[D, b_j]\right)^* \\
&= -A^* = -A
\end{aligned}
$$

Hence $A + JAJ^* = A - A = 0$, and the 'internal perturbation' $= 0$.

In the case of gauge theories over commutative algebras, one constructs connections on a principal fiber bundle where the structure group is a Lie group G. The particle fields are then sections of the associated vector bundle with fibers in $\mathcal{H}$. We thus have the following short exact sequence, where $\mathrm{Aut}(P)$ is the group of automorphisms on P, namely the diffeomorphisms f: $P \to P$ such that $f(pg) = f(p)g$, $\forall g \in G$, $p \in P$, and $GA(P)$ is the gauge group:

$$
1 \to GA(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M) \to 1
$$

This sequence is remarkably similar to the one in (24), which suggests the following prescription for constructing spectral gauge theories:

1. Look for an algebra $\mathcal{A}$ such that $\mathrm{Inn}(\mathcal{A}) \simeq GA(P)$.
2. Construct a suitable spectral triple 'over' $\mathcal{A}$.
3. Compute the spectral action (23).

The result of applying such a procedure would be a gauge theory of the group G coupled with gravity induced by the diffeomorphism group $\mathrm{Out}(\mathcal{A})$.

We end these remarks with comments on the problem of *spectral invariance versus diffeomorphism invariance*. If we denote by $spec(M, D)$ the spectrum of the Dirac operator with each eigenvalue repeated according to its multiplicity, then two manifolds M and M' are called *isospectral* if $spec(M, D) = spec(M', D)$. However, it is well known that there are manifolds which are isospectral without being isometric. Thus, spectral invariance is stronger than the usual diffeomorphic invariance and the eigenvalues of the Dirac operator cannot be used to distinguish among such manifolds ("one cannot hear the 'shape' of the drum") (Kac, 1996; Milnor, 1964).

There seems to be missing a paradigm that would allow us to distinguish among these manifolds; would it come from physics itself?

5.1. The Spectral Action for the Einstein–Yang–Mills System

The simplest noncommutative modification of a manifold M consists in a product of two spectral triples, one of which is the canonical triple discussed in the previous section and the other of which is associated with an N-dimensional matrix algebra $M_N(\mathbb{C})$. The spectral triple in which we are interested is then given by

$$\mathscr{A} = C^\infty(M) \otimes M_N(\mathbb{C}), \qquad \mathscr{H} = L^2(S, M) \otimes M_N(\mathbb{C}) \tag{34}$$

$$D_A = \gamma^\mu(\nabla_\mu \otimes 1_N - \frac{ig_0}{2} 1_4 \otimes A^i_\mu T^i)$$

where $A \in M_N(\mathbb{C})$ is a Hermitian matrix and T^i are the anti-Hermitian generators of the Lie algebra associated with the elements of a matrix group in $M_N(\mathbb{C})$. Making use of (17), we obtain the Lichnerowicz formula for the generalized Dirac operator D_A in the canonical decomposition:

$$D_A^2 = D^2 \otimes 1_N + \mathbf{E}, \tag{35}$$

where

$$\mathbf{E} = \frac{1}{4} R 1_4 \otimes 1_N + \frac{i}{8} g_0 \, [\gamma^\mu, \gamma^\nu] \, \partial_\mu A^i_\nu \otimes T^i \tag{36}$$

$$D^2 = \triangle^S + \tfrac{1}{4} R, \; \triangle^S = -g^{\mu\nu}(\nabla_\mu \nabla_\nu - \Gamma^\rho{}_{\mu\nu} \nabla_\rho),$$

where $\triangle^S$ is the Laplace–Beltrami operator associated to the spinor connection:

5.1.1. The Heat Kernel Expansion (Gilkey, 1984; Kalau and Walze, 1993)

Let $P = (D_A/\Lambda)^2$, where Λ is of the order of the inverse of the Planck length (10^{19} GeV). Using the functional Laplace transformation $\chi(P) = \int_0^\infty \exp(-tP)\rho(t)\, dt$, we have that

$$\mathrm{Tr}(\chi(P)) = \int_0^\infty \mathrm{Tr}(\exp\,(-tP))\, \rho(t)\, dt \tag{37}$$

Since we are assuming that M is compact, then P, which is positive, will have positive discrete eingenvalues and we can write $P\psi_n = \mu_n\psi_n$, $\mu_n \geq 0$. The “heat kernel”

$$G(x, y, t) = \sum_n \exp\,(-\mu_n t)\psi_n\,(x)\, \overline{\psi_n(y)} \tag{38}$$

is a solution to the heat equation $(P + \partial_t)G(x, y, t) = 0$. But from (38) and the completeness of the eigenfunctions, we get

$$\mathrm{Tr}\,\exp(-tP) = \int_M \sqrt{g(x)}\, G(x, x, t)\, d^4x$$

Using the asymptotic expansion for the kernel $G(x, x, t)$, one finds that

$$G(x, x, t) = \sum_{j=0}^\infty \Lambda^{4-2j} t^{j-2} a_{2j}(x, P) \tag{39}$$

$$\mathrm{Tr}\,\exp(-tP) = \sum_{j=0}^{\infty} \Lambda^{4-2j} t^{j-2} \underbrace{\int_M a_{2j}(x, P)\sqrt{g(x)}\, d^4x}_{a_{2j}(P)} \tag{40}$$

Substituting this expression into (37) yields

$$\mathrm{Tr}(\chi(P)) = \sum_{j=0}^{\infty} \Lambda^{4-2j}\, a_{2j}(P) \int_0^{\infty} t^{j-2}\, \rho(t)\, dt \tag{41}$$

Note also that

$$\int_0^{\infty} u\chi(u)\, du = \int_0^{\infty} \rho(t) \left(\int_0^{\infty} \exp\,(-tu)\, u\, du\right) dt$$

$$= \int_0^{\infty} t^{-2}\, \rho(t)\, dt =: f_0 \tag{42}$$

$$\int_0^{\infty} \chi(u)\, du = \int_0^{\infty} t^{-1}\, \rho(t)\, dt =: f_2 \tag{43}$$

$$(-1)^k \chi^k(0) = \int_0^{\infty} t^k\, \rho(t)\, dt =: f_{2(k+2)}, \qquad k \geq 0$$

Taking χ as a smoothened at $u = 1$ characteristic functional of the unit interval, so that

$$\chi(u) = 1, \quad u \leq 1; \qquad \chi(u) = 0, \quad u > 1; \qquad \chi^{(k)} = 0, \quad k \geq 0$$

we can integrate (42)–(43) to get $f_0 = \frac{1}{2}$, $f_2 = 1$, $f_4 = 1$. Thus, (41) becomes

$$S_B(D, A) = [\tfrac{1}{2}\Lambda^4 a_0(P) + \Lambda^2 a_2(P) + a_4(P)] \tag{44}$$

The scalar invariants a_{2k} can be read off from the work of Gilkey (1984) and de Witt (1965)

$$(4\pi)^2 a_0(P) = \int_M \sqrt{\det g}\, d^4x\, \mathrm{Tr}(1_4 \otimes 1_N) = N \int_M \sqrt{\det g}\, d^4x$$

$$(4\pi)^2 a_2(P) = \int_M \sqrt{\det g}\, \mathrm{Tr}\left(-\frac{R}{6}\, 1_4 \otimes 1_N + \mathbf{E}\right) d^4x \tag{45}$$

$$(4\pi)^2 a_4(P) = \frac{1}{360} \int_M \sqrt{\det g}\, \mathrm{Tr}[(-12R_{;\mu}{}^{;\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu}$$

$$+ 2\, R_{\mu\nu\sigma\rho}\, R^{\mu\nu\sigma\rho})1_4 \otimes 1_N$$

$$- 60R\mathbf{E} + 180\mathbf{E}^2 + 60\mathbf{E}_{;\mu}{}^{;\mu} + 30\Omega_{\mu\nu}\Omega^{\mu\nu}]\, d^4x \tag{46}$$

$$\Omega_{\mu\nu} = \frac{1}{8} R^{\alpha\beta}{}_{\mu\nu}\, [\gamma_\alpha, \gamma_\beta] \otimes 1_N - \frac{i}{2}\, 1_4 \otimes g_0\, \partial_\mu A^i_\nu \otimes T^i \tag{47}$$

For our final result, we only need to insert (36) and (47) into (45) and (46), recall that the T^i are traceless, make use of the Gauss–Bonnet topological invariant

$$\left(\frac{1}{2}\, \epsilon^{\mu\nu\rho\sigma}\, R^{\alpha\beta}{}_{\mu\nu}\right)\left(\frac{1}{2}\, \epsilon_{\alpha\beta\lambda\tau}\, R^{\lambda\tau}{}_{\rho\sigma}\right) = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

and substitute the derived expressions for the a_{2j} into (44) to get

$$\begin{aligned}\frac{48\pi^2}{N}\, S_B(D, A) = {}& 6\Lambda^4 \int_M d^4x\, \sqrt{g(x)} + \Lambda^2 \int_M d^4x \sqrt{g(x)} R \\ & + \int_M d^4x \sqrt{g(x)} \left(-\frac{3}{20}\, C_{\mu\nu\sigma\rho}\, C^{\mu\nu\sigma\rho} + \frac{1}{10}\, R_{;\mu}{}^{;\mu} \right. \\ & + \frac{11}{120} \left(\frac{1}{2}\, \epsilon^{\mu\nu\rho\sigma}\, R^{\alpha\beta}{}_{\mu\nu}\right)\left(\frac{1}{2}\, \epsilon_{\alpha\beta\lambda\tau}\, R^{\lambda\tau}{}_{\rho\sigma}\right) \\ & \left. + \frac{g_0^2}{N}\, g^{\mu\sigma}\, g^{\nu\rho}\, \partial_\mu A^i_\nu\, \partial_\sigma A^i_\rho \right)\end{aligned} \tag{48}$$

where $C_{\mu\nu\sigma\rho}$ is the Weyl tensor

$$C_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} - (g_{\mu|\sigma}R_{\nu|\rho]} - g_{\nu|\sigma}R_{\mu|\rho]}) + \tfrac{1}{6}\, (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})R$$

The Einstein–Yang–Mills theory with the action given in (48) contains the inverse-length-scale Λ cutoff, and the functional χ is chosen in such a way that it is equal to one when the eigenvalues of D_A^2/Λ^2 are near zero and equal to zero as these eigenvalues approach one. This reflects the implicit assumption that for distances smaller than the Planck length or, equivalently, for energies larger than the Planck energy, the manifold structure of space-time breaks down and one has to consider a possibly noncommutative algebra instead of $C^\infty(M)$ in the first factor of the product algebra (34). So the cutoff scale Λ is a regularizer of the theory, while the χ removes the nonrenormalizable interactions. Comparing the action with the conventional Euclidean form for the Einstein–Yang–Mills terms of the Lagrangian, we get

$$\frac{N\Lambda^2}{48\pi^2} = \frac{1}{16\pi G_0} \equiv \frac{1}{2\kappa_0^2}, \qquad \frac{g_0^2}{48\pi^2} = \frac{1}{4}$$

$$a_0 = \frac{-3N}{80}\frac{1}{g_0^2}, \qquad c_0 = -\frac{2}{3}a_0, \qquad d_0 = -\frac{11}{3}a_0, \qquad e_0 = \frac{N\Lambda^4}{8\pi^2}$$

$$S_B(D, A) = \int_M d^4x\,\sqrt{g(x)}\left[\frac{1}{2\kappa_0^2}R + e_0 + a_0 C_{\mu\nu\sigma\rho}C^{\mu\nu\sigma\rho} + c_0 R_{;\mu}^{;\mu}\right.$$
$$\left. + d_0\left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}^{\alpha\beta}\right)\left(\frac{1}{2}\epsilon_{\alpha\beta\lambda\tau}R_{\rho\sigma}^{\lambda\tau}\right) + \frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\,\partial_\mu A_\nu^1\,\partial_\sigma A_\rho^i\right)\right] \tag{49}$$

The renormalized Lagrangian, resulting from adding counterterms of the same form as those in the original action and subsequent rescaling of the parameters and fields, will contain the physical parameters κ, a, c, d, but will also have the additional term $\int_M d^4x\,\sqrt{g(x)}\,(bR^2)$. This system was studied in a different context by Fradkin and Tseytlin (1982) and shown to be renormalizable, while Stelle (1977) showed that it did not satisfy unitarity due to the presence of tachyons near the Planck length. However, this breakdown is to be expected and provides a justification for the ultimate need of considering noncommutative algebras.

We have considered this, grantedly, not too realistic model, first, because it is the simplest based on a noncommutative algebra, second, because it exhibits in a rather tractable way the details of the construction procedure outlined at the end of Section 5.1, and third, because it also exhibits the unifying features of spectral theory and some of its predictive powers, as, for example, the correct relative signs of the different terms appearing in the Lagrangian. As a parenthetical remark, note that the action (49) is dominated by the first term, which is a huge cosmological constant. This could pose serious objections, but as Landi and Roveli (1996) observed, there are tricks that one can use to eliminate such a term, for instance, by replacing the function χ by $\overline{\chi}$,

$$\overline{\chi}(u) = \chi(u) - a\chi(bu), \qquad a = b^2, \quad b \geq 0, \quad b \neq 1 \tag{50}$$

Then

$$\tilde{f}_0 = 0, \qquad \tilde{f}_2 = \left(1 - \frac{a}{b}\right)f_2, \qquad \tilde{f}_{2(k+2)} = (-1)^k(1 - ab^k)\chi^{(k)}(0), \qquad k > 0 \tag{51}$$

A more realistic case, namely the Einstein–Standard Model System, involves adding the different families of quarks and leptons to the above discussed theory. This in turn involves a much more complicated spectral triple, and its details are far beyond the space limitations for this presentation. Thus, for example, the Dirac operator for this theory will be a 45×45 matrix (a 36×36 matrix for the action on the 36 quarks, plus a 9×9 matrix for the action on leptons). We merely point out that the universal spectral action for the Standard Model coupled to gravitation is encoded in the formula

$$\mathrm{Tr}\left[\chi\left(\frac{D_A^2}{\Lambda^2}\right)\right] + (\psi, D\psi) \tag{52}$$

where the second term obviously describes the fermion interaction. For a detailed analysis of the mathematics and physics involved, see the original papers of Connes and Chamseddine (1996) as well as to the more thorough (and therefore more tractable in the calculations) papers of Kastler and coworkers (Carminati *et al.*, 1996; Iochum *et al.*, 1995, 1996). Here I prefer to devote a few last paragraphs to some of the outstanding problems and future outlook of spectral geometry as related to the ultimate goal of grand unification theories.

6. PROBLEMS AND FUTURE OUTLOOK

Below distances of the order of the Planck length, the concept of localization loses operational meaning and leads to a uncertainty in the structure of space-time, which is incorporated in noncommutative geometry (Connes, 1995; Carminati *et al.*, 1996; Doplicher *et al.*, 1995). Below the energy scale Λ, we trust the continuum approximation, so noncommutative geometry is a manifestation of gravity at the small scale. We have shown that gauge fields appear naturally as inner automorphisms of a unital noncommutative C^*-algebra; so in this context, the small-scale structure of space-time manifests itself also in the physics at energies lower than the Planck mass. The universal spectral action (52), together with the construction scheme discussed at the end of Section 5.2, would be the starting point to investigate this program. A major step forward would be the identification of the algebra $\mathcal{A}$ that would play the role of the gauge group in the noncommutative scenario. Spectral gravity has been developed so far in the context of Riemannian compact space-times. Wick rotations and $3 + 1$ splitting remain to be understood (Kalau, 1994).

We have used the case of the simplest noncommutative modifications of a manifold M, given by $\mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C})$, to show how the spectral action leads to a renormalizable (although nonunitary) theory for the Einstein–

Yang–Mills system. We also pointed out how, by introducing fermionic fields, this procedure has been applied to model the more realistic case of the Einstein–Standard Model system. It is interesting to observe that even though in this formalism the external geometry is still represented by an algebra of smooth functions in the first factor of the tensor algebra, there are already some important implications and predictions resulting from the theory. Thus, in the universal spectral action theory, the Higgs comes for free, and appears as a gauge field. With the assumption of $\Lambda \simeq 10^{15}$ GeV to fix the parameters of the Standard Model, the predicted mass of the Higgs scalar particle turns out to be somewhere within the range $160 < m_H < 200$ GeV. The spectral action associated with the Standard Model is consistent with the experimental data provided one takes $\Lambda \sim 10^{15}$ GeV [the unification scale for an SU(5) theory, for instance], and there is a 10% disagreement between the predicted and the experimental value of $\sin^2\theta_W$. Also, the relative signs of the different terms that appear in the action are the correct ones. However, taking $\Lambda \simeq 10^{15}$ GeV to fix the parameters of the Standard Model implies a very large value for the Newton constant in the gravity sector (which would require that $\Lambda \simeq 10^{19}$ GeV). The same problem occurs with string theory!

There are two critical issues that may lie behind the above disagreements. One has to do with the so called Big Desert issue, which assumes that between the present experimental range of $\sim 10^3$ GeV and the unification energy $\sim 10^{15}$ GeV of the electroweak–strong interactions, there is no new physics. That is, the Big Desert conjecture presupposes that the Standard Model remains valid without modification and there are no new particles, in particular.

The second issue has to do with the assumption that perturbation quantum field theory, which so far is our only available computational tool, gets through the Big Desert without collapsing. Are these two tremendous extrapolations from the experimental range sensible? The history of physics and the radical changes in our conception of nature, particularly those originated at the beginning of the 20th century, have taught us otherwise.

There is another intriguing idea that has been pursued by Connes, Moscovici, and Kreimer (Connes and Moscovici, 1998; Kreimer, 1998; Connes and Kreimer, 1998; Connes and Kreimer, 1999), and has to do with the fact that even after the radical changes introduced by quantum mechanics and general relativity in our picture of space-time, it still has some Kantian remnants. Perhaps we should think instead that the geometry of space-time is dictated by quantum field theory. By dualizing Riemannian geometry, we had $ds = G^{1/2} = |D|^{-1}$ [(9), (13)]. The inverse of the Dirac operator is the Feynman bare propagator in quantum field theory, and regularization followed by renormalization invariably implies a transition from the bare to the dressed propagator. This, according to Connes and Kreimer (1999), emphasizes the fact that space-time ought to be regarded as a derived concepts whose structure should follow from the properties of quantum field theory.

A remarkable result which appears to give support to the above contention is the observation that two Hopf algebras—one discovered by Connes and Moscovici (1998) in noncommutative geometry and the other discovered by Kreimer (Kreimer, 1998; Connes and Kreimer, 1998; Connes and Kreimer, 1999; Wulkenhaar, 1999) in the context of quantum field theory—are related, and that the antipode of this Hopf algebra reproduces precisely the combinatorics of renormalization (see Rosenbaum and Vergara, 2000 for further discussion).

It is too early to tell to what extent this program of noncommutative geometry will change our picture of black holes, big bang cosmology, and the origin of time. But of one thing we can be fairly certain, there are many "peels of the physics onion" left to be removed before claiming that, as some of our illustrious predecessors did at the end of the 19th century, there is no more fundamental physics left to be discovered!

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